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Approved for Distribution 7/11/63

SAFETY STOCK ALLOCATIONS

In Working Memorandum No. 128 a discussion is given of the problem of

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SAFETY STOCK ALLOCATIONS

In Working Memorandum No. 128 a discussion is given of the problem of finding the optimal exponent γ when the safety stock is taken to be proportional to the γ power of the mean demand. A solution is obtained under the assumption that the distribution of the positive-excess demand (the demand in excess of the mean demand in a lead time) is a power function, with exponent approximately - 1.

There are two conditions on a distribution function that are not met by this assumption. First, the function must be no greater than 1.0; in fact, in this case it must be no greater than F_o, the probability that the demand exceeds the mean in a lead time. Second, the expected value of the square must exist and be finite, since we want to assume that the standard deviation of demand in a lead time is a x^b , where x is the mean yearly demand. Of course, we can modify the power function (by truncation or otherwise) to meet these requirements, but the problem then becomes more difficult and the solution more complex. The solution is so complex,for any admissible distribution function that I have tried (and this includes several modifications of the power function), that it is not possible at this time to say anything about the form of the solution, except that b is an important parameter, and probably the most important one.

Corrections to Working Memorandum No. 128

The last line on page 4 should read:

$$S_1 \doteq (N\eta/K)(I_S/aN)^{\epsilon}(\bar{x})^{-\epsilon}\mu^{-\epsilon\,\gamma(\gamma-1)}(\bar{x})^{\epsilon(\gamma-b)+\frac{1}{2}\mu^{\epsilon^2(\gamma-b)^2}-1/4}$$

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This is obtained by using the integral approximation

$$S_1 \doteq \frac{N}{K} \int_0^{\infty} \sqrt{x} \;\; F(k(x))g(x)dx$$

for the sum, where F is the power function, k(x) is given by the last line on page 3, and g(x) is the log-normal density function for the distribution of the yearly demand among items.

The Power Function

Let u be the demand in a lead time and let

$$m = m(x)$$

$$\sigma = \sigma(x) = a\ x^b$$

be respectively the mean demand in a lead time and the standard deviation from the mean for an item having yearly mean demand x. The reorder point for this item is

$$r = m + \lambda\ x^{\gamma}$$

Since λ is positive and x is non-negative, a stockout can't occur when $u \leq m$. Hence, only the part of the distribution in excess of m is of interest.

To avoid writing F_o, the probability that the demand exceeds the mean, explicitly as a factor, consider the conditional upper distribution function H(y), given that the demand exceeds the mean, where

$$y = u - m.$$

Then, for $y \geq 0$, $H(y)$ is monotonic non-increasing with

$$H(0) = 1, \quad H(\infty) = 0$$

and

(1) $$\sigma^2 = \int_0^\infty y^2 \, dH(y)$$

Here, σ is the standard deviation from $y = 0$, since this value of y corresponds to $u = m$; and it is taken only on the positive side of the mean, since only these values of u are pertinent.

Assume that $H(y)$ is a function $F(k)$ of

$$k = y/\sigma$$

By this is meant that when y is replaced by $k\sigma$ the distribution function depends only on k, not on σ. Then $F(0) = 1$, $F(\infty) = 0$, and (1) becomes

(2) $$1 = \int_0^\infty k^2 \, dF(k)$$

Otherwise, F is a function of both k and σ.

In Working Memorandum No. 128 it is assumed that

$$F(k) = \eta k^{\epsilon}, \quad \epsilon < 0,$$

where η and ϵ are independent of σ. This is not an admissible function, since it is greater than 1.0 for $k < k_1$, where $\eta k_1^{\epsilon} = 1$.

By truncation we get the modified function

$$(3) \qquad F(k) = \begin{cases} 1 & , k < k_1 \\ \eta k^{\epsilon} & , k \geq k_1 \end{cases}$$

which has the density function

$$(4) \qquad f(k) = \begin{cases} 0 & , k < k_1 \\ -\eta \epsilon k^{\epsilon - 1} & , k > k_1 \end{cases}$$

The integral in (2) exists only if $\epsilon < -2$, unless we want to truncate also at some value of $k = k_2$. The function in (3) with $\epsilon < -2$ is an admissible distribution function. However, it is not acceptable as a reasonable distribution, since it is not likely that there will be no demands near the mean, as indicated by (4). In fact, one would expect that this region would have highest density. Also, the integral approximation for S_1 now would be much more complicated, since it would be a sum of two integrals, each of which involves an error integral.

A Modified Power Function

An acceptable modification of the power function is

$$f(k) = \begin{cases} f_1 & , 0 \leq k \leq k_2 \\ -\eta \epsilon k^{\epsilon - 1} & , k > k_2 \end{cases}$$

where

$$f_1 = -\eta \epsilon \ k_2^{\epsilon - 1} \ , \quad \epsilon < -2.$$

Then

$$F(k) = \begin{cases} 1 - f_1 k & , 0 \leq k \leq k_2 \\ \eta k^{\epsilon} & , k > k_2 \end{cases}$$

and

$$1 - f_1 k_2 = \eta k_2^{\epsilon} = - f_1 k_2/\epsilon$$

or

$$f_1 k_2 (1 - 1/\epsilon) = 1.$$

Another condition on the parameters is given by (2).

Now

$$k = k(x) = c\, x^{\gamma - b} ,$$

where

$$c = c(\gamma) = (I_S/aN)(\bar{x})^{-\gamma} \mu^{-\gamma(\gamma - 1)}$$

In terms of x the distribution function is

$$F(k(x)) = \begin{cases} 1 - f_1 c\, x^{\gamma-b} & , \ 0 \leq x \leq x_2 \\ \eta c^{\epsilon}\, x^{\epsilon(\gamma-b)} & , \ x > x_2 \end{cases}$$

where

$$k_2 = c(\gamma)x_2^{\gamma-b}$$

The integral approximation of S_1 becomes

$$S_1 \doteq \frac{N}{K}\left[\int_0^{x_2} x^{1/2}\,(1 - f_1 c x^{\gamma-b})\, g(x)\, dx + \eta c^{\epsilon} \int_{x_2}^{\infty} x^{1/2+\epsilon(\gamma-b)} g(x) dx\right]$$

If g(x) is the log-normal density function, S_1 can be expressed in closed form in terms of the error integral. Similarly, the expected amount S_2 of the shortage can be expressed in closed form. But the forms will be so complicated that there is little hope of obtaining the minimums analytically. A solution for any given set of parameters could be found by approximation.

Rectangular Distribution

The only admissible distribution F(k) that I have been able to treat in a semi-analytical manner, when g(x) is the log-normal density function, is the rectangular distribution,

$$F(k) = \begin{cases} 1 - k/\sqrt{3} & , \quad 0 \leq k \leq \sqrt{3} \\ 0 & , \quad k > \sqrt{3} \end{cases}$$

The $\sqrt{3}$ factor is needed to satisfy (2); that is, starting with $f(k) = 1/k_1$ for $0 \leq k \leq k_1$, and 0 elsewhere, equation (2) yields $k_1 = \sqrt{3}$. Then

$$S_1 = \frac{N}{K} \int_{x_1}^{x_2} x^{1/2} \left[1 - c\, x^{\gamma - b} / \sqrt{3} \right] g(x)\, dx \ ,$$

where (x_1, x_2) is the interval of x for which

$$0 \leq c x^{\gamma - b} \leq \sqrt{3}.$$

If $\gamma > b$, $x_1 = 0$ and $c\, x_2^{\gamma - b} = \sqrt{3}$. If $\gamma < b$, $c\, x_1^{\gamma - b} = \sqrt{3}$, and $x_2 = \infty$.

For $\gamma > b$ the expression for S_1 becomes

$$S_1 = (N/K)\ddot{x}^{1/2} \mu^{-1/4} \left[\Phi\left(\frac{1}{s} \log \rho \right) - \mu^{(\gamma - b)^2} \rho^{-(\gamma - b)} \Phi\left(\frac{1}{s} \log \rho - s(\gamma - b) \right) \right]$$

where

$$s = \sqrt{2 \log \mu}$$

$$\rho = x_2/\bar{x} = (1/\bar{x})(\sqrt{3}/c)^{1/(\Upsilon-b)} \quad , \; c = c(\Upsilon)$$

$$\Phi(u) = (1/\sqrt{2\pi}) \int_{-\infty}^{u} e^{-t^2/2}\, dt$$

For $\Upsilon < b$, the Φ functions are replaced by their complements.

The derivative is

$$\frac{dS_1}{d\Upsilon} = \begin{cases} A\left[1+c_1\sqrt{2\pi}\;\exp(z^2/2)\;\Phi(z)\right] , & \Upsilon \geq b \\ A\left[-1+c_1\sqrt{2\pi}\;\exp(z^2/2)(1-\Phi(z))\right] , & \Upsilon < b, \end{cases}$$

where $A > 0$ and

(5) $$z = c_1 + c_2/w - w/2,$$

(6) $$w = s(\Upsilon - b) \;,$$

$$c_1 = s(b-1/2)$$

$$c_2 = \log(Na\sqrt{3}/I_s) + b \log \bar{x} + b(b-1) \log \mu$$

We can find the value of z that minimizes S_1 and then substitute in (5) and (6) to find Υ.

The results are as follows:

(a) If $c_2 < 0$, there is no finite solution. In this case, I_s is so large that it does not impose an essential restriction.

(b) If $c_2 = 0$, $\Upsilon = b$

(c) If $c_2 > 0$, the form of the solution depends on the sign of c_1.

(1) If $c_1 = 0$, that is, $b = 1/2$,

then $\Upsilon = b = 1/2$

(2) If $c_1 \neq 0$

$$\Upsilon = \begin{cases} b - \frac{1}{s}\left(\sqrt{J^2 + 2c_2} - J\right), & b > 1/2 \\ b + \frac{1}{s}\left(\sqrt{J^2 + 2c_2} - J\right), & b < 1/2 \end{cases}$$

where

$$J = J(|c_1|) = |z - c_1|$$

is graphed in Figure 1. To find Υ, compute the value of $|c_1| = |b - 1/2|$ s read the value of J from Figure 1, and substitute in the above formulas.

If $4c_2 \ll |c_1|$, Υ is approximated by

$$\Upsilon \doteq \begin{cases} b - c_2/sJ, & b > 1/2 \\ b + c_2/sJ, & b < 1/2 \\ b, & b = 1/2 \end{cases}$$

From: James M. Dobbie/rc

Arthur D. Little, Inc.

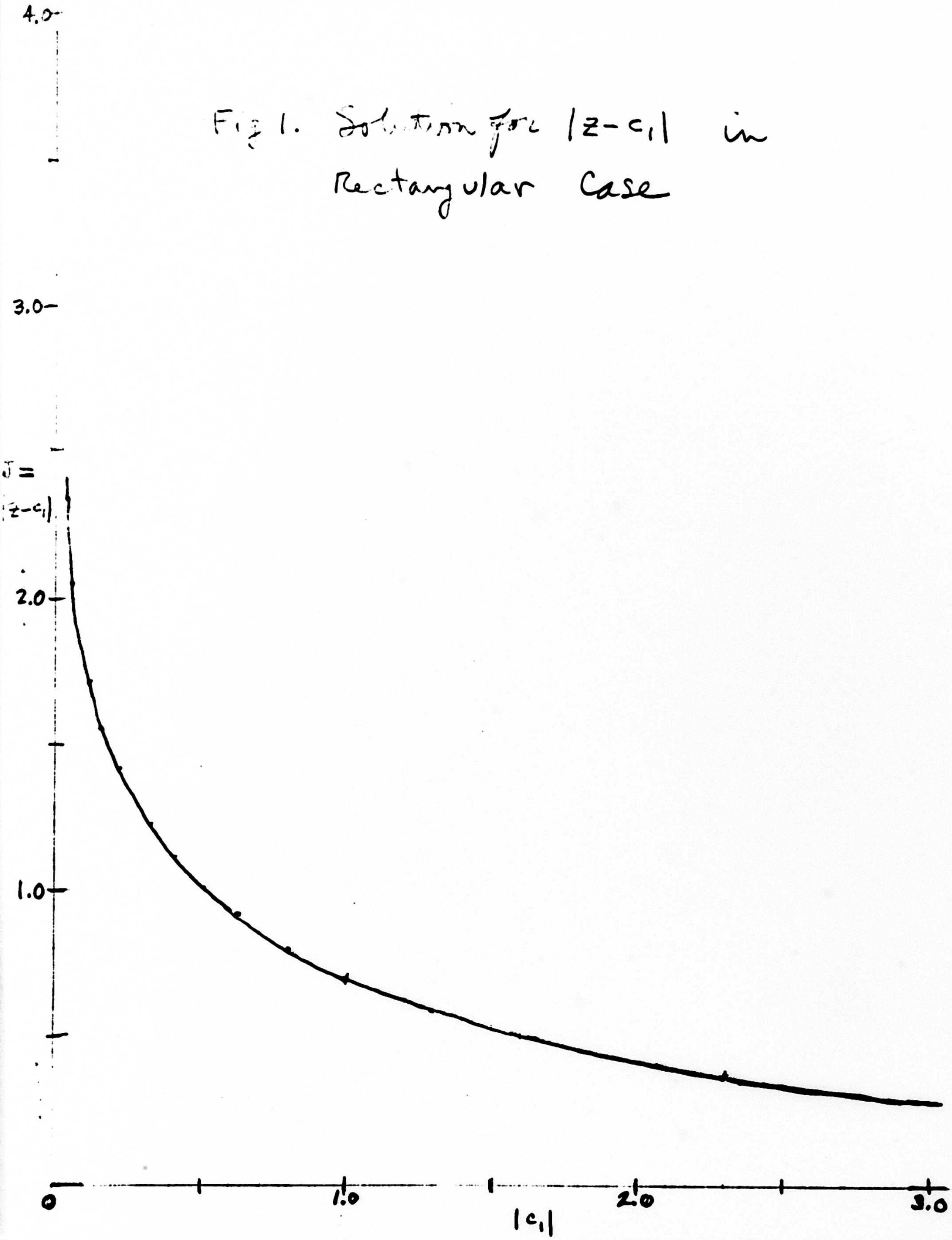

Fig 1. Solution for $|z - c_1|$ in Rectangular Case